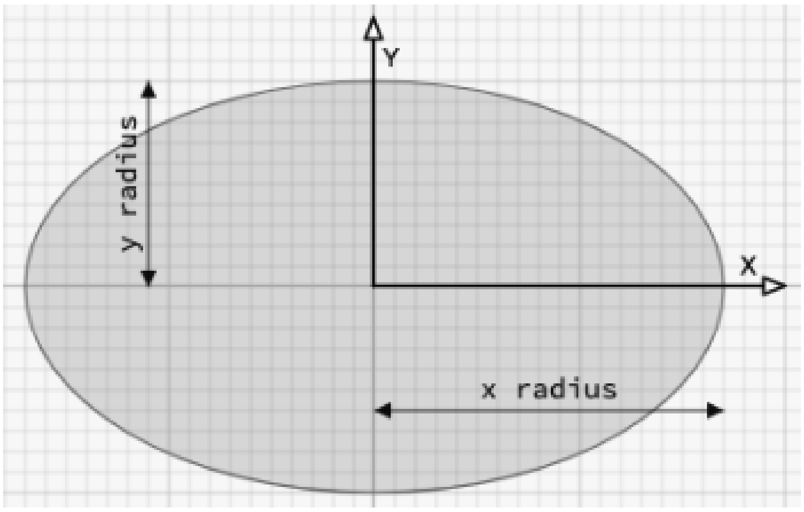
Intersections of two arbitrary ellipses.

by Peter Lager (2023)

Abstract

This article describes how to calculate the closed form solution for the intersections of two arbitrary ellipses. It starts with the standard equation for an ellipse and takes the reader through the algebra to create the implicit equation of a rotated and translated ellipse. With the implicit equations for two ellipses the reader is guided through the method needed to solve the simultaneous equations and hence find any intersections.

1 The implicit equation of an ellipse



The equation of an axis aligned ellipse whose centre is at $[0, 0]$ is given by the formula -

$$\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} = 1 \tag{1}$$

To represent an arbitrary ellipse the equation needs to include the effects of rotation and translation. The order we apply these operations is important as they are not commutative and in this scenario we will

1. rotate the ellipse about its centre
2. move the ellipse centre by translation.

This will create an implicit equation of the form -

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

1.1 Rotation

Rotating a point $[x, y]$ by $\boldsymbol{\theta}$ about the origin $[0, 0]$ transforms the point to $[x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta}, x\sin\boldsymbol{\theta} + y\cos\boldsymbol{\theta}]$ which can be substituted into (1)

$$\begin{aligned}
\frac{(x\cos\boldsymbol{\theta} - y\sin\boldsymbol{\theta})^2}{r_x^2} - \frac{(x\sin\boldsymbol{\theta} + y\cos\boldsymbol{\theta})^2}{r_y^2} &= 1 \\
\frac{(x^2\cos^2\boldsymbol{\theta} - 2\,xy\cos\boldsymbol{\theta}\sin\boldsymbol{\theta} + \sin^2\boldsymbol{\theta}\, y^2)^2}{r_x^2} + \frac{(x^2\sin^2\boldsymbol{\theta} + 2\,xy\cos\boldsymbol{\theta}\sin\boldsymbol{\theta} + y^2\cos^2\theta)^2}{r_y^2} - 1 &= 0 \\
\left(\frac{\cos^2\boldsymbol{\theta}}{r_x^2} + \frac{\sin^2\boldsymbol{\theta}}{r_y^2}\right)x^2 + \left(\frac{-2\cos\boldsymbol{\theta}\sin\boldsymbol{\theta}}{r_x^2} + \frac{2\cos\boldsymbol{\theta}\sin\boldsymbol{\theta}}{r_y^2}\right)xy + \left(\frac{\sin^2\boldsymbol{\theta}}{r_x^2} + \frac{\cos^2\boldsymbol{\theta}}{r_y^2}\right)y^2 - 1 &= 0
\end{aligned}$$

then simplify by letting
$A=\left(\frac{\cos^2\boldsymbol{\theta}}{r_x^2}+\frac{\sin^2\boldsymbol{\theta}}{r_y^2}\right)$, $B=\left(\frac{-2\cos\boldsymbol{\theta}\sin\boldsymbol{\theta}}{r_x^2}+\frac{2\cos\boldsymbol{\theta}\sin\boldsymbol{\theta}}{r_y^2}\right)$, $C=\left(\frac{\sin^2\boldsymbol{\theta}}{r_x^2}+\frac{\cos^2\boldsymbol{\theta}}{r_y^2}\right)$ and $F=-1$

So the implicit formula for a rotated ellipse becomes

$$Ax^2+B\,\mathrm{xy}+\mathrm{Cy}^2+F \;=\; 0 \tag{2}$$

1.2 Translation

Substituting $[x-c_x, y-c_y]$ into (2) shifts the centre of the ellipse to $[c_x, c_y]$.

$$\begin{aligned}
A\,(x-c_x)^2+B(x-c_x)\,(y-c_y)+C(y-c_y)^2+F &= 0\\
A(x^2-2x\,c_x+c_x^2)+B(x\,y-x\,c_y-y\,c_x+c_x\,c_y)-C(y^2-2y\,c_y+c_y^2)+F &= 0\\
(A)x^2+(B)x\,y+(C)y^2-(2A\,c_x+Bc_y)x-(Bc_x+2Cc_y)y+(Ac_x^2+Bc_xc_y+Cc_y^2+F) &= 0
\end{aligned}$$

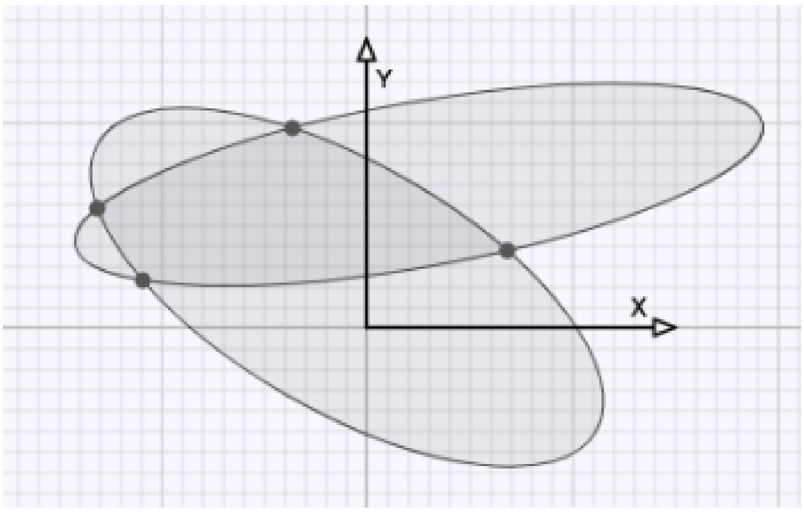
We can simplify the equation using the substitutions

$$\begin{aligned}
a &= A\\
b &= B\\
c &= C\\
d &= Bc_x+2Cc_y\\
e &= Bc_x+2Cc_y\\
f &= Ac_x^2+Bc_xc_y+Cc_y^2+F
\end{aligned}$$

These identities provide the coefficients for the implicit equation of an arbitrary ellipse (3) centered on $[c_x, c_y]$ and a rotation angle of $\boldsymbol{\theta}$.

$$a\,x^2+b\,x\,y+c\,y^2+d\,x+e\,y+f=0 \tag{3}$$

2 Calculating the intersections



Consider two arbitrary ellipses s and t with the implicit equations

$$\begin{aligned}
a_s\,x^2+b_s\,x\,y+c_s\,y^2+d_s\,x+e_s\,y+f_s &= 0 \qquad (4)\\
a_t\,x^2+b_t\,x\,y+c_t\,y^2+d_t\,x+e_t\,y+f_t &= 0 \qquad (5)
\end{aligned}$$

2.1 Solving for y

The next step is solve the simultaneous equations (4) and (5) for y to get the explicit function $x=g(y)$.

First eliminate the x^2 term by $(4).a_t - (5).a_s$

$$(b_s.a_t - b_t.a_s)xy + (c_s.a_t - c_t.a_s)\,y^2 + (d_s.a_t - d_t.a_s)\,x + (e_s.a_t - e_t.a_s)\,y + (f_s.a_t - f_t.a_s) \;=\; 0 \tag{6}$$

Let $b = (b_s.a_t - b_t.a_s)$, $c = (c_s.a_t - c_t.a_s)$, $d = (d_s.a_t - d_t.a_s)$, $e = (e_s.a_t - e_t.a_s)$ and $f = (f_s.a_t - f_t.a_s)$ and substitute into (6) to get

$$b.x.y + c.y^2 + d.x + e.y + f \;=\; 0 \tag{7}$$

Rearrange (7) to get the explicit function $x = g(y)$

$$x \;=\; \frac{-(c.y^2 + e.y + f)}{(b.y + d)} \tag{8}$$

2.2 Find the coefficents for the quartic of y

Let $m = c.y^2 + e.y + f$ and $n = b.y + d$ and substitute into (8) to get $x = -m/n$. Substitute $x = -m/n$ into (4) and multiply by n^2

$$\begin{aligned}
\frac{a_s.m^2}{n^2} - \frac{b_s.m}{n}y + c_s\,y^2 - \frac{d_s.m}{n} + e_s\,y + f_s &= 0 \\
a_s.m^2 - b_s.m.n.y + c_s.n^2.y^2 - d_s.m.n + e_s.n^2.y + f_s.n^2 &= 0 \\
a_s.m^2 + (c_s.y^2 + e_s.y + f_s)n^2 - (b_s.y + d_s).m.n &= 0 \qquad (9)
\end{aligned}$$

Expand m^2, n^2 and $m.n$

$$\begin{aligned}
m^2 &= (c.y^2 + e.y + f).(c.y^2 + e.y + f) \\
&= c^2.y^4 + 2.c.e.y^3 + (2.c.f + e^2).y^2 + 2.e.f.y + f^2 \qquad (10) \\
n^2 &= (b.y + d).(b.y + d) \\
&= b^2.y^2 + 2.b.d.y + d^2 \qquad (11) \\
m.n &= (c.y^2 + e.y + f).(b.y + d) \\
&= b.c.y^3 + (c.d + b.e).y^2 + (d.e + b.f).y + d.f \qquad (12)
\end{aligned}$$

Let $p = (c.d + b.e)$ and $q = (d.e + b.f)$ and substitute into (12)

$$m.n \;=\; b.c.y^3 + p.y^2 + q.y + d.f \tag{13}$$

Substitute (10), (11) and (13) into (9) and expand each term

$$\begin{aligned}
\text{Term 1} &= a_{s.m^2} \\
&= a_s.(c^2.y^4 + 2.c.e.y^3 + (2.c.f + e^2).y^2 + 2.e.f.y + f^2) \\
&= a_s.c^2.y^4 + 2.a_s.c.e.y^3 + a_s.(2.c.f + e^2).y^2 + 2.a_s.e.f.y + a_s.f^2
\end{aligned}$$

$$\begin{aligned}
\text{Term 2} &= (c_s.y^2 + e_s.y + f_s)n^2 \\
&= (c_s.y^2 + e_s.y + f_s).(b^2.y^2 + 2.b.d.y + d^2) \\
&= c_s.b^2.y^4 + (2.c_s.b.d + e_s.b^2).y^3 + (c_s.d^2 + 2.e_s.b.d + f_s.b^2)y^2 + (e_s.d^2 + 2.f_s.b.d).y + f_s.d^2
\end{aligned}$$

$$\begin{aligned}
\text{Term 3} &= -(b_s.y + d_s).m.n \\
&= -(b_s.y + d_s).(b.c.y^3 + p.y^2 + q.y + d.f) \\
&= -b_s.b.c.y^4 - (d_s.b.c + b_s.p).y^3 - (d_s.p + b_s.q).y^2 - (d_s.q + b_s.d.f).y - d_s.d.f
\end{aligned}$$

Collect the terms to calculate the coefficients for the quartic

$$z_4.y^4 + z_3.y^3 + z_2.y^2 + z_1.y + z_0 = 0$$

$$\begin{aligned}
z_4 &= a_s.c^2 + c_s.b^2 - b_s.b.c \\
z_3 &= 2.a_s.c.e + (2.c_s.b.d + e_s.b^2) - (d_s.b.c + b_s.p) \\
z_2 &= a_s.(2.c.f + e^2) + (c_s.d^2 + 2.e_s.b.d + f_s.b^2) - (d_s.p + b_s.q) \\
z_1 &= 2.a_s.e.f + (e_s.d^2 + 2.f_s.b.d) - (d_s.q + b_s.d.f) \\
z_0 &= a_s.f^2 + f_s.d^2 - d_s.d.f
\end{aligned}$$

3 Solving the quartic

$$z_4.y^4 + z_3.y^3 + z_2.y^2 + z_1.y + z_0 = 0 \tag{14}$$

A quartic equation (14) has 4 roots where some or all of them may be complex numbers. For this problem only real roots are needed and complex roots can be ignored. If all the roots are complex numbers then the ellipses do not interesct.

Like lower order polynomials the quartic equation has a closed-form (algebraic) solution but it is more challenging because

- it is neccessary to solve cubic and sometimes quadratic equations as part of the algorithm.
- even if all the roots are real the intermediate calculations involve complex numbers.

The algorithms used in solving a quartic are beyond the scope of this document but I hope to include it in another article at a later date.

$$\text{Let} \quad y_0\,,y_1\,,y_2 \ \text{ and } \ y_3 \quad \text{be the roots of (14)}$$

Any imaginary roots can be ignored but for each real root it is necessary to calculate the matching x coordinate. Since any real root (y_n) must satisfy the implicit equations for both ellipses (4) and (5) then x_n can be found by substituting y_n into equation (8) to get

$$x_n \;=\; \frac{-(c.y_n^2 + e.y_n + f)}{(b.y_n + d)} \tag{15}$$

If both ellipses are centred about the y-axis (i.e. $c_x = 0$) and neither have been rotated then the denominator $(b.y_n + d)$ will be zero and x_n undefined. In this case x_n can be obtained by the solution of the quadratic -

$$\begin{aligned} a x_n^2 + b x_n + c &= 0 \\ \text{where} \qquad\qquad a &= s_a \\ b &= s_b\, y_n + s_d \\ c &= s_c\, y_n^2 + s_e\, y_n + f \end{aligned}$$

This can be solved usuing the well known formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The result sepends on the determinant $(b^2 - 4ac)$

$$\begin{aligned} b^2 - 4ac \ &< 0 \ \ \text{no intersection so ignore} \\ &= 0 \ \ \text{single intersection } [x_n, y_n] \\ &> 0 \ \ \text{two intersections } [x_n, y_n] \ \text{ and } \ [-x_n, y_n] \end{aligned}$$

Any real roots would provide two intersections $[x_n, y_n]$ and $[-x_n, y_n]$.

4 Programming a graphical solution

I have used these equations to create a Javascript program showing the intersections between multiple animated ellipses. The program uses p5.js to provide a simple framework to display javascript apps in a webpage.

The equations used in this article are based on the standard cartesian coordinate system but in computer graphics the y axis is inverted. To cope with this the program has to invert the rotation so in section 1.1 the angle $\boldsymbol{\theta}$ should be replaced with $-\boldsymbol{\theta}$.

The other issue to consider is the accuracy of floating-point-numbers (fpn) and fpn calculations in a computer.

In the last section it was neccessary to see if the denominator $(b.y_n + d)$ was zero, if Javascript we might have the statement

```
if(denominator == 0){
    // x is undefined so create a quadratic solution
}
```

The condition is true *only* if the `denominator` is *exactly* zero, but computers store fpn with a precsion limited by the data type. Many fpn calculations were performed to calculate the value of `denominator` therefore it is quite possible that `denominator = 0.000000000007` even though in theory it should be zero. The solution is to compare the calculated value with the test value if the difference between the two is less than a predetermined limit, assume they are the same.

In Javascript it might look like this

```
if(Math.abs(denominator) < EPSILON){
    // x is undefined so create a quadratic solution
}
```

Where the constant `EPSILON` has a very small value e.g. 10^{-10}.

5 Conclusion

This article does not cover how to calculate the roots of the quartic equation but there are online calculators that are easy to use, this calculator comes from Casio. Once you have the real roots it is easy to find the corresponding x values and hence the intersections.

Programming a complete grahical solver would require the program to

- store and perfom basic maths with complex numbers.
- solve quartic, cubic and quadratic equations with real and complex roots.

and I hope to cover these in later articles.